Finding Equations of Tangents and Normals

Key points:

- Gradient of tangent = $m = \frac{dy}{dx}$, Equation of tangent: $y - y_1 = m(x - x_1)$
- Gradient of normal = $m_n = \frac{-1}{m}$, Equation of normal: $y - y_1 = m_n(x - x_1)$

Examples:

1) Find the equation of the tangent to the curve given by $y = x^2 + 6x - 2$ at (2, 14).

$m = \frac{dy}{dx} = 2x + 6$, $m(2) = 2(2) + 6 = 10.$

Eqn of tangent: $y - 14 = 10(x - 2)$

$y - 14 = 10x - 20$

$y = 10x - 6$

2) Find the equation of the normal to the curve given by $y = 4x - x^2$ when x = -1.

$m = \frac{dy}{dx} = 4 - 2x$, $m(-1) = 4 - 2(-1) = 6$, $m_n = \frac{-1}{6}$

When $x = -1$, $y = 4x - x^2 = 4(-1) - (-1)^2 = -5$

Eqn of normal: $y - (-5) = \frac{-1}{6}(x - (-1))$

$y + 5 = \frac{-1}{6}x - \frac{1}{6}$

$y = \frac{-1}{6}x - 5\frac{1}{6}$

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Using implicit differentiation

3) Use implicit differentiation to calculate the equation of the tangent to the circle $x^2 + y^2 = 100$ at the point (-8, 6).

$x^2 + y^2 = 100$ differentiates to $2x + 2y \cdot \frac{dy}{dx} = 0.$

Rearrange to make $\frac{dy}{dx}$ the subject: $\frac{dy}{dx} = \frac{-x}{y}$

$m = \frac{dy}{dx} = \frac{-x}{y}$, $m(-8, 6) = \frac{-(-8)}{6} = \frac{4}{3}$

Eqn of tangent: $y - 6 = \frac{4}{3}(x - (-8))$

$y - 6 = \frac{4}{3}x + \frac{32}{3}$

$y = \frac{4}{3}x + \frac{50}{3}$

4) Differentiate $xy^2 + x = 4$, and calculate the equation of the normal at the point (2, 1).

$x \cdot y^2 + x = 4$ differentiates to $\left(1 \cdot y^2 + 2y\frac{dy}{dx} \cdot x\right) + 1 = 0$, remembering to use the product rule when differentiating $x \cdot y^2$ implicitly.

Rearrange to make $\frac{dy}{dx}$ the subject: $\frac{dy}{dx} = \frac{-y^2-1}{2xy}$

$m(2, 1) = \frac{-1^2-1}{2(2)(1)} = \frac{-2}{4} = \frac{-1}{2}$

$m_n(2, 1) = \frac{-1}{-1/2} = 2$

Eqn of normal: $y - 1 = 2(x - 2)$

$y - 1 = 2x - 4$

$y = 2x - 3$

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<u>Using parametric differentiation</u>

5) The parametric equations $\begin{Bmatrix} x = (t+1)^3 \\ y = t^2 - 2 \end{Bmatrix}$ define a curve. Calculate the equations of the tangent and the normal to the curve at the point (8, -1).

$\frac{dy}{dt} = 2t \qquad \frac{dx}{dt} = 3 \cdot (t+1)^{3-1} \cdot 1 = 3(t+1)^2$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{3(t+1)^2} = \frac{2t}{3(t+1)^2}$

We need a 't' value for the coordinate (8, -1):

When $x = 8$, $(t+1)^3 = 8 \ \rightarrow \ t + 1 = \sqrt[3]{8} \ \rightarrow \ t + 1 = 2 \ \rightarrow \ t = 1$

When $y = -1, t^2 - 2 = -1 \ \rightarrow \ t^2 = 1 \ \rightarrow \ t = \pm 1 \ \rightarrow \ t = 1$

$m(1) = \frac{2(1)}{3(1+1)^2} = \frac{2}{12} = \frac{1}{6} \qquad m_n = \frac{-1}{1/6} = -6$

Eqn of tangent at point (8, -1): $y - (-1) = \frac{1}{6}(x - 8)$

$y + 1 = \frac{1}{6}x - 1\frac{1}{3}$

$y = \frac{1}{6}x - 2\frac{1}{3}$

Eqn of normal at point (8, -1): $y - (-1) = -6(x - 8)$

$y + 1 = -6x + 48$

$y = -6x + 47$

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